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RICCI SOLITONS IN CONTACT METRIC MANIFOLDS

Mukut Mani Tripathi

СОЛИТОНЫ РИЧЧИ В КОНТАКТНЫХ МЕТРИЧЕСКИХ МНОГООБРАЗИЯХ

М. М. Трипатхи

In $N(k)$-contact metric manifolds and/or (k,μ)-manifolds, gradient Ricci solitons, compact Ricci solitons and Ricci solitons with V pointwise collinear with the structure vector field ξ are studied.

В работе изучаются солитоны Риччи, в $N(k)$-контактных метрических многообразиях и в контактных (k,μ)-многообразиях

Keywords: солитоны Риччи, $N(k)$-контактные метрические многообразия, (k,μ)-многообразия, K-контактные многообразия, многообразия Сасаки.

Ключевые слова: Ricci soliton, $N(k)$-contact metric manifold, (k,μ)-manifold, K-contact manifold, Sasakian manifold.

1. Introduction

A Ricci soliton is a generalization of an Einstein metric. In a Riemannian manifold (M,g), g is called a Ricci soliton [18] if

$$\pounds_V g + 2\,Ric + 2\lambda g = 0, \qquad (1)$$

where $\pounds$ is the Lie derivative, V is a complete vector field on M and λ is a constant. Metrics satisfying (1) are interesting and useful in physics and are often referred as quasi-Einstein (e.g. [9], [10], [15]). Compact Ricci solitons are the fixed point of the Ricci flow

$$\frac{\partial}{\partial t}\, g = -\,2\,Ric$$

projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. The Ricci soliton is said to be shrinking, steady, and expanding according as λ is negative, zero, and positive respectively. If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and equation (1) assumes the form

$$\nabla\nabla f = Ric + \lambda g. \qquad (2)$$

A Ricci soliton on a compact manifold has constant curvature in dimension 2 (Hamilton [18]), and also in dimension 3 (Ivey [19]). For details we refer to Chow and Knoff [12] and Derdzinski [14]. We also recall the following significant result of Perelman [24]: *A Ricci soliton on a compact manifold is a gradient Ricci soliton.*

On the other hand, the roots of contact geometry lie in differential equations as in 1872 Sophus Lie introduced the notion of contact transformation (Berührungstransformation) as a geometric tool to study systems of differential equations. This subject has manifold connections with the other fields of pure mathematics, and substantial applications in applied areas such as mechanics, optics, phase space of a dynamical system, thermodynamics and control theory (for more details see [1], [3], [16], [21] and [22]).

It is well known [26] that the tangent sphere bundle T_1M of a Riemannian manifold M admits a contact metric structure. If M is of constant curvature $c = 1$ then T_1M is Sasakian [33], and if $c = 0$ then the curvature tensor R satisfies $R(X,Y)\xi = 0$ [2]. As a generalization of these two cases, in [5], Blair, Koufogiorgos and Papantoniou started the study of the class of contact metric manifolds, in which the structure vector field ξ satisfies the (k,μ)-nullity condition. A contact metric manifold belonging to this class is called a (k,μ)-manifold. Such a structure was first obtained by Koufogiorgos [20] by applying a D_a-homothetic deformation [?] on a contact metric manifold satisfying $R(X,Y)\xi = 0$. In particular, a $(k,0)$-manifold is called an $N(k)$-contact metric manifold ([4], [6], [32]) and generalizes the cases $R(X,Y)\xi = 0$, K-contact and Sasakian.

In [28], Sharma has initiated the study of Ricci solitons in K-contact manifolds. In a K-contact manifold the structure vector field ξ is Killing, that is, $\pounds_\xi g = 0$; which is not in general true in contact metric manifolds. Motivated by these circumstances, in this paper we study Ricci solitons in $N(k)$-contact metric manifolds and (k,μ)-manifolds. In section , we give a brief description of $N(k)$-contact metric manifolds and (k,μ)-manifolds. In section , we prove main results. Among others, we prove that in a non-Sasakian (or non-K-contact) $N(k)$-contact metric manifold (M,g), if the metric g is a Ricci soliton with V pointwise collinear with ξ, then $\dim(M) > 3$, the metric g is a shrinking Ricci soliton and M is locally isometric to a contact metric manifold obtained by a $D_{\left(1+\frac{(\sqrt{n}\pm1)^2}{n-1}\right)}$-homothetic deformation of the contact metric structure on the tangent sphere bundle of an $(n+1)$-dimensional Riemannian manifold of constant curvature $\frac{(\sqrt{n}\pm1)^2}{n-1}$.

2. Contact metric manifolds

A 1-form η on a $(2n+1)$-dimensional smooth manifold M is called a *contact form* if $\eta\wedge(d\eta)^n \neq 0$ everywhere on M, and M equipped with a contact form is a *contact manifold*. For a given contact 1-form η, there exists a unique vector field ξ, called the *characteristic vector field*, such that $\eta(\xi) = 1$, $d\eta(\xi,\cdot) = 0$, and consequently $\pounds_\xi\eta = 0$, $\pounds_\xi d\eta = 0$. In 1953, Chern [11] proved that the structural group of a $(2n+1)$-dimensional contact manifold can be reduced to $\mathcal{U}(n)\times 1$. A $(2n+1)$-dimensional differentiable manifold M is called an *almost contact manifold* [17] if its structural group can be reduced to $\mathcal{U}(n)\times 1$. Equivalently, there is an *almost contact structure* (φ,ξ,η) [25] consisting of a tensor field φ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying

$$\varphi^2 = -I + \eta\otimes\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta\circ\varphi = 0. \tag{3}$$

First and one of the remaining three relations of (3) imply the other two relations. An almost contact structure is *normal* [27] if the torsion tensor $[\varphi,\varphi] + 2d\eta\otimes\xi$, where $[\varphi,\varphi]$ is the Nijenhuis tensor of φ, vanishes identically. Let g be a compatible Riemannian metric with (φ,ξ,η), that is,

$$g(X,Y) = g(\varphi X,\varphi Y) + \eta(X)\eta(Y), \; X,Y \in TM. \tag{4}$$

Then, M becomes an *almost contact metric manifold* equipped with an *almost contact metric structure* (φ,ξ,η,g). The equation (4) is equivalent to

$$g(X,\varphi Y) = -g(\varphi X,Y) \quad \text{alongwith} \quad g(X,\xi) = \eta(X). \tag{5}$$

An almost contact metric structure becomes a contact metric structure if $g(X,\varphi Y) = d\eta(X,Y)$ for all $X,Y \in TM$. In a contact metric manifold M, the $(1,1)$-tensor field h defined by $2h = \pounds_\xi\varphi$, is symmetric and satisfies

$$h\xi = 0, \qquad h\varphi + \varphi h = 0, \tag{6}$$

$$\nabla\xi = -\varphi - \varphi h, \tag{7}$$

where ∇ is the Levi-Civita connection. A contact metric manifold is called a *K-contact manifold* if the characteristic vector field ξ is a Killing vector field. An almost contact metric manifold is a K-contact manifold if and only if $\nabla\xi = -\varphi$. A K-contact manifold is a contact metric manifold, while the converse is true if $h = 0$. A normal contact metric manifold is a *Sasakian manifold*. A contact metric manifold M is Sasakian if and only if the curvature tensor R satisfies

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \quad X,Y \in TM. \tag{8}$$

A contact metric manifold M is said to be *η-Einstein* ([23] or see [3] p. 105) if the Ricci tensor Ric satisfies $Ric = ag + b\eta\otimes\eta$, where a and b are some smooth functions on the manifold. In particular if $b = 0$, then M becomes an *Einstein manifold*.

A Sasakian manifold is always a K-contact manifold. The converse is true if either the dimension is three ([3], p. 76), or it is compact Einstein (Theorem A, [8]) or compact η-Einstein with $a > -2$ (Theorem 7.2, [8]). The conclusions of Theorems A and 7.2 of [8] are still true if the condition of compactness is weakened to completeness (Proposition 1, [28]).

In [5], Blair, Koufogiorgos and Papantoniou introduced a class of contact metric manifolds M, which satisfy

$$R(X,Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y), \tag{9}$$

where k, μ are real constants. A contact metric manifold belonging to this class is called a *(k,μ)-manifold*. If $\mu = 0$, then a (k,μ)-manifold is called an *$N(k)$-contact metric manifold* ([4], [6], [32]). In a (k,μ)-manifold M, one has [5]

$$(\nabla_X h)Y = ((1-k)\,g(X,\varphi Y) + g(X,\varphi hY))\,\xi + \eta(Y)(h(\varphi X + \varphi hX)) - \mu\eta(X)\varphi hY \tag{10}$$

for all $X,Y \in TM$. The Ricci operator Q satisfies $Q\xi = 2nk\xi$, where $\dim(M) = 2n+1$. Moreover, $h^2 = (k-1)\varphi^2$ and $k \leq 1$. In fact, for a (k,μ)-manifold, the conditions of being a Sasakian manifold, a K-contact manifold, $k = 1$ and $h = 0$

are all equivalent. The tangent sphere bundle T_1M of a Riemannian manifold M of constant curvature c is a (k,μ)-manifold with $k = c(2-c)$ and $\mu = -2c$. Characteristic examples of non-Sasakian (k,μ)-manifolds are the tangent sphere bundles of Riemannian manifolds of constant curvature not equal to one and certain Lie groups [7]. For more details we refer to [3] and [5].

3. Main results

Let (M,φ,ξ,η,g) be a $(2n+1)$-dimensional non-Sasakian (k,μ)-manifold. Then the Ricci operator Q is given by [5]

$$Q = 2nkI+ + (2(n-1)+\mu)h - (2(n-1)-n\mu+2nk)\varphi^2. \quad (11)$$

We also have

$$(\nabla_X\varphi^2)Y = = (X\eta(Y))\xi - \eta(\nabla_X Y)\xi - \eta(Y)\varphi X - \eta(Y)\varphi hX, \quad (12)$$

where first equation of (3) and equation (7) are used. Using (10) and (12) from (11) we obtain

$$\begin{aligned}(\nabla_X Q)Y &= (2(n-1)+\mu)\{(1-k)g(X,\varphi Y)\xi + g(X,\varphi hY)\xi - \mu\eta(X)\varphi hY\}\\ &\quad - (2(n-1)-n\mu+2nk)\{(X\eta(Y))\xi - \eta(\nabla_X Y)\xi\}\\ &\quad + (2(2n-1)k-(n+1)\mu+k\mu)\eta(Y)\varphi X + ((n+1)\mu-2nk)\eta(Y)h\varphi X.\end{aligned}$$

Consequently, we have

$$\begin{aligned}(\nabla_X Q)Y - (\nabla_Y Q)X &= (2(n+1)\mu - 4(2n-1)k - 2k\mu)\,d\eta(X,Y)\xi\\ &\quad + (2(2n-1)k-(n+1)\mu+k\mu)(\eta(Y)\varphi X - \eta(X)\varphi Y)\\ &\quad + ((\mu+3n-1)\mu-2nk)(\eta(Y)\varphi hX - \eta(X)\varphi hY),\end{aligned} \quad (13)$$

where (5) has been used.

We also recall the following results for later use.

Theorem 3.1. (Theorem 5.2, Tanno [32]) *An Einstein $N(k)$-contact metric manifold of dimension ≥ 5 is necessarily Sasakian.*

Theorem 3.2. (Theorem 1.2, Tripathi and Kim [34]) *A non-Sasakian Einstein (k,μ)-manifold is flat and 3-dimensional.*

Now we prove the following

Theorem 3.3. *If the metric g of an $N(k)$-contact metric manifold (M,g) is a gradient Ricci soliton, then*

- **(a)** *either the potential vector field is a nullity vector field,*
- **(b)** *or g is a shrinking soliton and (M,g) is Einstein Sasakian,*
- **(c)** *or g is a steady soliton and (M,g) is 3-dimensional and flat.*

Proof. Let (M,g) be a $(2n+1)$-dimensional $N(k)$-contact metric manifold and g a gradient Ricci soliton. Then the equation (2) can be written as

$$\nabla_Y Df = QY + \lambda Y \quad (14)$$

for all vector fields Y in M, where D denotes the gradient operator of g. From (14) it follows that

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X,\ X,Y \in TM. \quad (15)$$

We have

$$g(R(\xi,Y)Df,\xi) = g(k(Df-(\xi f)\xi),Y), \quad (16)$$

where (9) with $\mu = 0$ is used. Also in an $N(k)$-contact metric manifold, it follows that

$$g((\nabla_\xi Q)Y - (\nabla_Y Q)\xi,\xi) = 0, \qquad Y \in TM. \quad (17)$$

From (15), (16) and (17) we get

$$k(Df-(\xi f)\xi) = 0,$$

that is, either $k = 0$ or

$$Df = (\xi f)\xi. \quad (18)$$

If $k = 0$, then putting $k = 0 = \mu$ in (13), it follows that Q is a Codazzi tensor, that is,

$$(\nabla_X Q)Y - (\nabla_Y Q)X = 0, \qquad X,Y \in TM,$$

which in view of (15) gives

$$R(X,Y)Df = 0, \qquad X,Y \in TM,$$

that is, the potential vector field Df is a nullity vector field (see [13] and [31] for details).

Now, we assume that (18) is true. Using (18) in (14) we get

$$Ric(X,Y) + \lambda g(X,Y) = = Y(\xi f)\eta(X) - (\xi f)g(X,\varphi Y) - (\xi f)g(X,\varphi hY),$$

where (7) is used. Symmetrizing this with respect to X and Y we obtain

$$2\,Ric\,(X,Y) + 2\lambda g\,(X,Y) =$$

$$= X\,(\xi f)\,\eta\,(Y) + Y\,(\xi f)\,\eta\,(X) - 2\,(\xi f)\,g\,(\varphi hX,Y)\,. \tag{19}$$

Putting $Y = \xi$, we get

$$X\,(\xi f) = (2nk + \lambda)\,\eta\,(X)\,. \tag{20}$$

From (19) and (20) we get

$$Ric\,(X,Y) + \lambda g\,(X,Y) =$$

$$= (2nk + \lambda)\,\eta\,(X)\,\eta\,(Y) - (\xi f)\,g\,(\varphi hX,Y)\,. \tag{21}$$

Using (21) in (14), we get

$$\nabla_Y Df = (2nk + \lambda)\,\eta\,(Y)\,\xi - (\xi f)\,\varphi hY. \tag{22}$$

Using (22) we compute $R\,(X,Y)\,Df$ and obtain

$$g\,(R\,(X,Y)\,(\xi f)\,\xi,\xi) = 4\,(2nk + \lambda)\,d\eta\,(X,Y)\,, \tag{23}$$

where equations (18) and (7) are used. Thus we get

$$2nk + \lambda = 0 \tag{24}$$

Therefore from equation (20) we have

$$X\,(\xi f) = 0, \qquad X \in TM,$$

that is,

$$\xi f = c,$$

where c is a constant. Thus the equation (18) gives

$$df = c\,\eta\,.$$

Its exterior derivative implies that

$$c\,d\eta = 0,$$

that is, $c = 0$. Hence f is constant. Consequently, the equation (14) reduces to

$$Ric = -\lambda g = 2nkg,$$

that is, M is Einstein. Then in view of Theorem 3.2 and Theorem 3.1, it follows that either M is Sasakian or M is 3-dimensional and flat. In case of Sasakian, $\lambda = -2n$ is negative, and therefore the soliton g is shrinking. In case of 3-dimensional and flat, $\lambda = 0$, and therefore the soliton g is steady.

Corollary 3.4. *Let (M,g) be a compact $N(k)$-contact metric manifold with $k \neq 0$. If g is a Ricci soliton, then g is a shrinking soliton and (M,g) is Einstein Sasakian.*

Proof. The proof follows from Theorem 3.3 and the following significant result of Perelman [24]: A Ricci soliton on a compact manifold is a gradient Ricci soliton.

In [28], a corollary of Theorem 1 is stated as follows: If the metric g of a compact K-contact manifold is a Ricci soliton, then g is a shrinking soliton which is Einstein Sasakian. In Corollary , the assumptions are weakened.

Next, we have the following

Theorem 3.5. *In a non-Sasakian (k,μ)-manifold (M,g) if g is a compact Ricci soliton, then (M,g) is 3-dimensional and flat.*

Proof. In a non-Sasakian (k,μ)-manifold, the scalar curvature r is given by [5]

$$r = 2n\,(2n - 2 + k - n\mu)\,. \tag{25}$$

Consequently, the scalar curvature is a constant. If g is a compact Ricci soliton, then by Proposition 2 of [28], which states that a compact Ricci soliton of constant scalar curvature is Einstein, it follows that the non-Sasakian (k,μ)-manifold is Einstein. Then by Theorem 3.2, it becomes 3-dimensional and flat.

Given a non-Sasakian (κ,μ)-manifold M, Boeckx [7] introduced an invariant

$$I_M = \frac{1 - \mu/2}{\sqrt{1-\kappa}}$$

and showed that for two non-Sasakian (κ,μ)-manifolds $(M_i,\varphi_i,\xi_i,\eta_i,g_i)$, $i = 1,2$, we have $I_{M_1} = I_{M_2}$ if and only if up to a D-homothetic deformation, the two manifolds are locally isometric as contact metric manifolds. Thus we know all non-Sasakian (κ,μ)-manifolds locally as soon as we have for every odd dimension $2n+1$ and for every possible value of the invariant I, one (κ,μ)-manifold (M,φ,ξ,η,g) with $I_M = I$. For $I > -1$ such examples may be found from the standard contact metric structure on the tangent sphere bundle of a manifold of constant curvature c where we have $I = \frac{1+c}{|1-c|}$. Boeckx also gives a Lie algebra construction for any odd dimension and value of $I \leq -1$.

In the following, we recall Example 3.1 of [6].

Example 3.6. For $n > 1$, the Boeckx invariant for a $(2n+1)$-dimensional $\left(1 - \frac{1}{n}, 0\right)$-manifold is $\sqrt{n} > -1$. Therefore, we consider the tangent sphere bundle of an $(n+1)$-dimensional manifold of constant curvature c so chosen that the resulting D_a-homothetic deformation will be a $\left(1 - \frac{1}{n}, 0\right)$-manifold. That is for $k = c(2-c)$ and $\mu = -2c$ we solve

$$1 - \frac{1}{n} = \frac{k + a^2 - 1}{a^2}, \quad 0 = \frac{\mu + 2a - 2}{a}$$

for a and c. The result is

$$c = \frac{\left(\sqrt{n} \pm 1\right)^2}{n-1}, \quad a = 1 + c$$

and taking c and a to be these values we obtain a $N\left(1 - \frac{1}{n}\right)$-contact metric manifold.

In [28], Sharma noted that if a K-contact metric is a Ricci soliton with $V = \xi$ then it is Einstein.

Even in more general case, he showed that if a K-contact metric is a Ricci soliton with V pointwise collinear with ξ then V is a constant multiple of ξ (hence Killing) and g is Einstein. Here we prove the following

Theorem 3.7. *Let (M, g) be a non-Sasakian (or non-K-contact) $N(k)$-contact metric manifold. If the metric g is a Ricci soliton with V pointwise collinear with ξ, then $\dim(M) > 3$, the metric g is a shrinking Ricci soliton and M is locally isometric to a contact metric manifold obtained by a $D_{\left(1+\frac{(\sqrt{n}\pm1)^2}{n-1}\right)}$-homothetic deformation of the contact metric structure on the tangent sphere bundle of an $(n+1)$-dimensional Riemannian manifold of constant curvature $\frac{(\sqrt{n}\pm1)^2}{n-1}$.*

Proof. Let (M, g) be a $(2n+1)$-dimensional contact metric manifold and the metric g a Ricci soliton with $V = \alpha\xi$ (α being a function on M). Then from (1) we obtain

$$2Ric(X,Y) = -2\lambda g(X,Y) + 2\alpha g(\varphi hX, Y) - \\ -g((X\alpha)\xi, Y) - g(X, (Y\alpha)\xi), \quad (26)$$

where (7) and (5) are used. Now let (M, g) be an $N(k)$-contact metric manifold. Putting $X = \xi = Y$ in (26) and using $h\xi = 0$ and $Q\xi = 2nk$ we get

$$\xi\alpha + 2nk + \lambda = 0. \quad (27)$$

Again putting $X = \xi$ in (26) and using $h\xi = 0$, $Q\xi = 2nk$ and (27) we get

$$d\alpha = (2nk + \lambda)\eta, \quad (28)$$

which shows that α is a constant and $\lambda = -2nk$; and consequently (26) becomes

$$Ric(X,Y) = 2nkg(X,Y) + \alpha g(\varphi hX, Y). \quad (29)$$

At this point, we assume that (M, g) is also non-Sasakian. It is known that in a $(2n+1)$-dimensional non-Sasakian (k, μ)-manifold M the Ricci tensor is given by [5]

$$Ric(X,Y) = (2(n-1) - n\mu)g(X,Y) + \\ +(2(n-1) + \mu)g(hX,Y) + \\ + (2(1-n) + n(2k+\mu))\eta(X)\eta(Y). \quad (30)$$

Consequently, putting $\mu = 0$ in (30) we get

$$Ric(X,Y) = 2(n-1)g(X,Y) + \\ +2(n-1)g(hX,Y) + \\ +(2(1-n) + 2nk)\eta(X)\eta(Y). \quad (31)$$

Replacing X by φX in equations (29) and (31) and equating the right hand sides of the resulting equations we get

$$(2nk - 2(n-1))g(\varphi X, Y) = \\ = \alpha g(hX,Y) + 2(n-1)g(\varphi hX, Y). \quad (32)$$

If $n = 1$, from (32) we get

$$2kg(\varphi X, Y) = \alpha g(hX, Y),$$

which gives $h = 0$, a contradiction. If $n > 1$, anti-symmetrizing the equation (32) we get

$$nk - n + 1 = 0,$$

which gives $k = 1 - 1/n$. Using $n > 1$ and $k = 1 - 1/n$ in $\lambda = -2nk$, we get $\lambda = 2(1-n) < 0$, which shows that g is a shrinking Ricci soliton. Finally, in view of $n > 1$, $k = 1 - 1/n$ and the Example 3.6, the proof is complete.

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БАНАХОВО МНОГООБРАЗИЕ ИНТЕГРАЛЬНЫХ КРИВЫХ ВПОЛНЕ ПАРАЛЛЕЛИЗУЕМОЙ СИСТЕМЫ ПФАФФА

В. Н. Черненко

BANACH MANIFOLD OF INTEGRAL CURVES OF COMPLETELY PARALLELIZABLE PFAFFIAN SYSTEM

V. N. Chernenko

В данной работе изучается множество интегральных кривых вполне параллелизуемой системы Пфаффа. Показано, что это множество является банаховым многообразием.

In this paper we study the set of integral curves completely parallelizable Pfaffian system. It is shown that this set is a Banach manifold.

Ключевые слова: система Пфаффа, интегральные кривые, банахово многообразие.

Keywords: Pfaffian system, integral curves, Banach manifold.