УДК 515.162.8

REIDEMEISTER MOVES FOR KNOTS AND LINKS IN LENS SPACES

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ПРЕОБРАЗОВАНИЯ РЕЙДЕМЕЙСТЕРА ДЛЯ УЗЛОВ И ЗАЦЕПЛЕНИЙ В ЛИНЗОВЫХ ПРОСТРАНСТВАХ

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We extend the concept of diagrams and associated Reidemeister moves for links in $\mathbf{S}^3$ *to links in lens spaces, using a differential approach. As a particular case, we obtain diagrams and Reidemeister type moves for links in* $\mathbb{RP}^3$ *introduced by Y.V. Drobothukina.*

В данной работе понятия диаграммы и преобразований Рейдемейстера, известные для зацеплений в $\mathbf{S}^3$*, распространяются для зацеплений в линзовых пространствах. В частности, получены диаграммы и преобразования типа Рейдемейстера для зацеплений в* $\mathbb{RP}^3$*, введенные ранее Ю.В. Дроботухиной.*

Ключевые слова: преобразования Рейдемейстера, линзовые пространства, трехмерные многообразия.

Keywords: Reidemeister moves, lens spaces, 3-manifolds.

1. Preliminaries

In this paper we work in the *Diff* category (of smooth manifolds and smooth maps). Every result also holds in the *PL* category, and in the *Top* category if we consider only tame links.

Definition 1. Let X and Y be two smooth manifolds.

A smooth map $f : X \to Y$ is an *embedding* if the differential $d_x f$ is injective for all $x \in X$ and if X and $f(X)$ are homeomorphic. As a consequence, X and $f(X)$ are diffeomorphic and $f(X)$ is a submanifold of Y.

An *ambient isotopy* between two embeddings l_0 and l_1 from X to Y is a smooth map $H : Y \times [0, 1] \to Y$ such that, if we write at each $t \in [0, 1]$, $H(y, t) = h_t(y)$, then $h_t : Y \to Y$ is a diffeomorphism, $h_0 = \mathrm{Id}_Y$ and $l_1 = h_1 \circ l_0$.

Definition 2. (Links) A *link* in a closed 3-manifold M^3 is an embedding of ν copies of $\mathbf{S}^1$ into M^3, namely it is $l : \mathbf{S}^1 \sqcup \ldots \sqcup \mathbf{S}^1 \to M^3$. A link can also be denoted by L, where $L = l(\mathbf{S}^1 \sqcup \ldots \sqcup \mathbf{S}^1) \subset M^3$. A *knot* is a link with $\nu = 1$.

Two links L_0 and L_1 are *equivalent* if there exists an ambient isotopy between the two embeddings l_0 and l_1.

Definition 3. (Lens spaces) Let p and q be two integer numbers such that $\gcd(p, q) = 1$ and $0 \leqslant q < p$. Consider $B^3 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leqslant 1\}$ and let E_+ and E_- be respectively the upper and the lower closed hemisphere of ∂B^3. Call B_0^2 the equatorial disk, defined by the intersection of the plane $x_3 = 0$ with B^3. Label with N and S respectively the Tnorth poleУ $(0, 0, 1)$ and the Tsouth poleУ $(0, 0, -1)$ of B^3.

Let $g_{p,q} : E_+ \to E_+$ be the rotation of $2\pi q/p$ around the x_3 axis as in Figure 1, and let $f_3 : E_+ \to E_-$ be the reflection with respect to the plane $x_3 = 0$. The *lens space* $L(p, q)$ is the quotient of B^3 by the equivalence relation on ∂B^3 which identify $x \in E_+$ with $f_3 \circ g_{p,q}(x) \in E_-$. We denote by $F : B^3 \to B^3/\sim$ the quotient map. Notice that on the equator $\partial B_0^2 = E_+ \cap E_-$ there are p points in each class of equivalence.

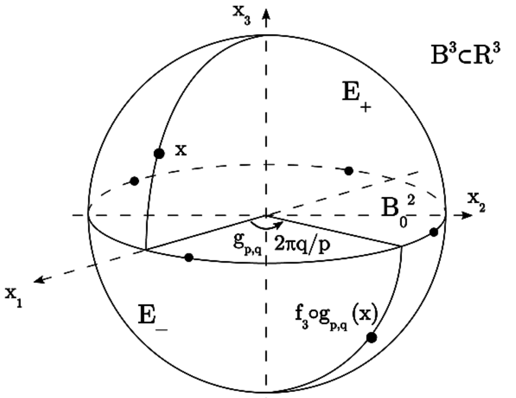

Fig. 1. Representation of $L(p, q)$

It is easy to see that $L(1, 0) \cong \mathbf{S}^3$ since $g_{1,0} = \mathrm{Id}_{E_+}$. Furthermore, $L(2, 1)$ is $\mathbb{RP}^3$, since we obtain the usual model of the projective space where opposite points of the boundary of the ball are identified.

Proposition 4. [1] *The lens spaces* $L(p, q)$ *and* $L(p', q')$ *are diffeomorphic (as well as homeomorphic) if and only if* $p' = p$ *and* $q' \equiv \pm q^{\pm 1}$*mod* p.

2. Links in $\mathbf{S}^3$

2.1. Diagrams

One of the first tools used to study links in $\mathbf{S}^3$ are diagrams, that is to say, a suitable projection of the links on a plane.

Definition 5. Let L be a link in $\mathbf{S}^3 = \mathbb{R}^3 \cup \{\infty\}$. Since L is compact, up to an affine transformation of $\mathbb{R}^3$, we can suppose that L belongs to $\mathrm{int}B^3$.

Let $\mathbf{p} : B^3 \smallsetminus \{N, S\} \to B_0^2$ be the projection defined in the following way: take $x \in B^3 \smallsetminus \{N, S\}$, construct the circle (or the line) $c(x)$ through N, x and S and set $\mathbf{p}(x) := c(x) \cap B_0^2$.

Take $L \subset \mathrm{int}B^3$ and project it using $\mathbf{p}_{|L} : L \to B_0^2$. For $P \in \mathbf{p}(L)$, $\mathbf{p}_{|L}^{-1}(P)$ may contain more than one point; in this case, we say that P is a *multiple point*. In particular, if it contains exactly two points, we say that P is a *double point*. We can assume, by moving L via a small isotopy, that the projection $\mathbf{p}_{|L} : L \to B_0^2$ of L is *regular*, namely:

1. the arcs of the projection contain no cusps;
2. the arcs of the projection are not tangent to each other;
3. the set of multiple points is finite, and all of them are actually double points.

These requests correspond to violations represented in Figure 2.

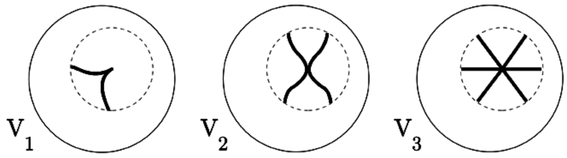

Fig. 2. Violations V_1, V_2 and V_3

Now let Q be a double point and consider $\mathbf{p}_{|L}^{-1}(Q) = \{P_1, P_2\}$. We suppose that P_2 is nearer to S than P_1. Take U as an open neighborhood of P_2 in L such that $\mathbf{p}(\overline{U})$ does not contain other double points. We call U *underpass*. Take the complementary set in L of all the underpasses. Every connected component of this set (as well as its projection in B_0^2) is called *overpass*. The underpasses are visualized in the projection by removing U from L' before projecting the link (see Figure 3). Observe that we may have components of the link which are single overpasses.

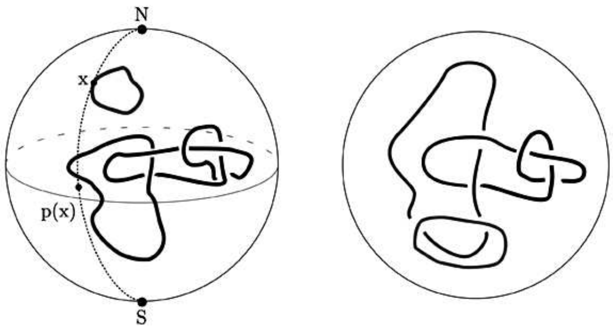

Fig. 3. A link in S^3 and corresponding diagram

Definition 6. A *diagram* of a link L in $\mathbf{S}^3$ is a regular projection of L on the equatorial disk B_0^2, with specified overpasses and underpasses.

2.2. Reidemeister moves

There are three (local) moves that allow us to determine when two links in $\mathbf{S}^3$ are equivalent from their diagrams. Reidemeister proved this theorem for *PL* links. For the *Diff* case a good reference is [7], where the proof involves links in arbitrary dimensions, so it is rather complicated.

Definition 7. The *Reidemeister moves* on a diagram of a link $L \subset \mathbf{S}^3$ are the moves R_1, R_2, R_3 of Figure 4.

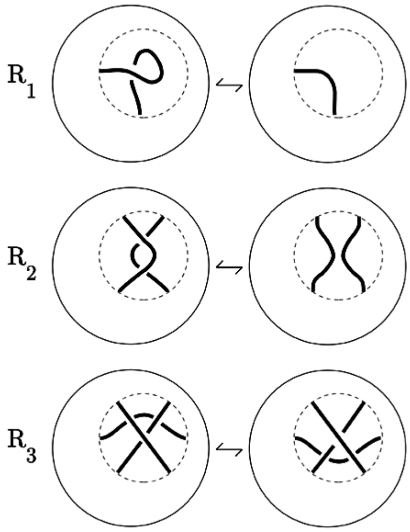

Fig. 4. Reidemeister moves

Theorem 8. [7] *Two links L_0 and L_1 in $\mathbf{S}^3$ are equivalent if and only if their diagrams can be joined by a finite sequence of Reidemeister moves R_1, R_2, R_3 and diagram isotopies.*

Proof. It is easy to see that each Reidemeister move produces equivalent links, hence a finite sequence of Reidemeister moves and isotopies on a diagram does not change the equivalence class of the link.

On the other hand, if we have two equivalent links L_0 and L_1, then we have an ambient isotopy, namely: $H : \mathbf{S}^3 \times [0,1] \to \mathbf{S}^3$, such that $l_1 = h_1 \circ l_0$. At each $t \in [0,1]$ we have a link L_t, defined by $h_t(l_0)$. Thanks to general position theory (see [7] for details), we can assume that the projection of L_t is not regular only a finite number of times, and that at each of these times it violates only one condition.

From each type of violation a transformation of the diagram appears, that is to say, a Reidemeister move, as it follows (see Figure 5):

– from violation V_1 we obtain move R_1;
– from violation V_2 we obtain move R_2;
– from violation V_3 we obtain move R_3.

So diagrams of two equivalent links can be joined by a finite sequence of Reidemeister moves R_1, R_2, R_3 and diagram isotopies. □

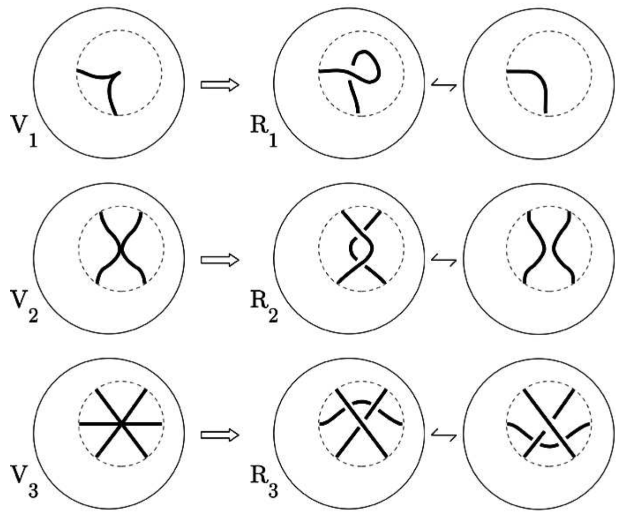

Fig. 5. Regularity violations produce Reidemeister moves

3. Links in $\mathbb{RP}^3$

3.1. Diagrams

The definition given by Drobothukina [3] of diagram for links in the projective space makes use of the model of the projective space $\mathbb{RP}^3$ explained in Section 1, as a particular case of $L(p,q)$ with $p = 2$ and $q = 1$. Namely, consider B^3 and identify diametrically opposed points on its boundary (let $\sim$ be the equivalence relation), so $\mathbb{RP}^3 \cong B^3/\sim$ and the quotient map is denoted by F.

Definition 9. Let L be a link in $\mathbb{RP}^3$. Consider $L' := F^{-1}(L)$; by moving L via a small isotopy in $\mathbb{RP}^3$, we can suppose that:

i) L' does not meet the poles S and N of B^3;

ii) $L' \cap \partial B^3$ consists of a finite set of points.

So L' is the disjoint union of closed curves in $\text{int}B^3$ and arcs properly embedded in B^3 (i.e. only the boundary points belong to ∂B^3).

Let $\mathbf{p} : B^3 \setminus \{N,S\} \to B^2_0$ be the projection defined in the following way: take $x \in B^3 \setminus \{N,S\}$, construct the circle (or the line) $c(x)$ through N, x and S and set $\mathbf{p}(x) := c(x) \cap B^2_0$.

Take L' and project it using $\mathbf{p}_{|L'} : L' \to B^2_0$. For $P \in \mathbf{p}(L')$, $\mathbf{p}^{-1}_{|L'}(P)$ may contain more than one point; in this case, we say that P is a *multiple point.* In particular, if it contains exactly two points, we say that P is a *double point.* We can assume, by moving L via small isotopies, that the projection $\mathbf{p}(L')$ is *regular*, namely:

1) the arcs of the projection contain no cusps;

2) the arcs of the projection are not tangent to each other;

3) the set of multiple points is finite, and all of them are actually double points;

4) the arcs of the projection are not tangent to ∂B^2_0;

5) no double point is on ∂B^2_0.

These requests correspond to violations represented in Figures 2 and 6.

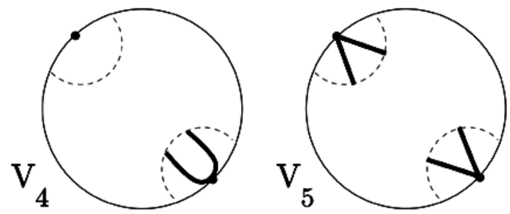

Fig. 6. Violations V_4 and V_5

Now let Q be a double point and consider $\mathbf{p}_{|L'}^{-1}(Q) = \{P_1, P_2\}$. We suppose that P_2 is nearer to S than P_1. Take U as an open neighborhood of P_2 in L' such that $\mathbf{p}(\overline{U})$ does not contain other double points and does not meet ∂B_0^2. We call U *underpass*. Take the complementary set in L' of all the underpasses. Every connected component of this set (as well as its projection in B_0^2) is called *overpass*. The underpasses are visualized in the projection by removing U from L' before projecting the link (see Figure 7).

Definition 10. A *diagram* of a link L in $\mathbb{RP}^3$ is a regular projection of $L' = F^{-1}(L)$ on the equatorial disk B_0^2, with specified overpasses and underpasses.

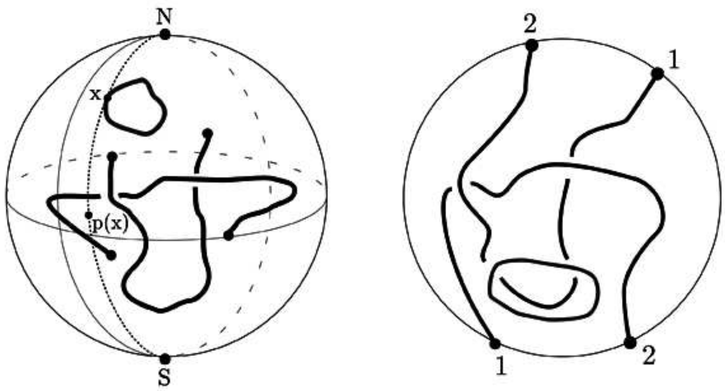

Fig. 7. A link in $L(2,1)$ and corresponding diagram

We label the boundary points of the link projection, in order to show the identifications. Assume that the equator is oriented counterclockwise if we look at it from N, and that the number of boundary points is $2t$. Choose a point of $\mathbf{p}(L')$ on the equator and call it 1 as well as the antipodal point, then following the orientation of ∂B_0^2, label the points of $\mathbf{p}(L')$ on the equatorial circle, as well as the antipodal ones, $2, \ldots, t$ (see Figure 7).

3.2. Generalized Reidemeister moves

We want to look for an analogue of the Reidemeister moves for links in $\mathbf{S}^3$, in order to understand when two diagrams of links in $\mathbb{RP}^3$ represent equivalent links.

Definition 11. The *generalized Reidemeister moves* on a diagram of a link $L \subset \mathbb{RP}^3$ are the moves R_1, R_2, R_3 of Figure 4 and the moves R_4, R_5 of Figure 8.

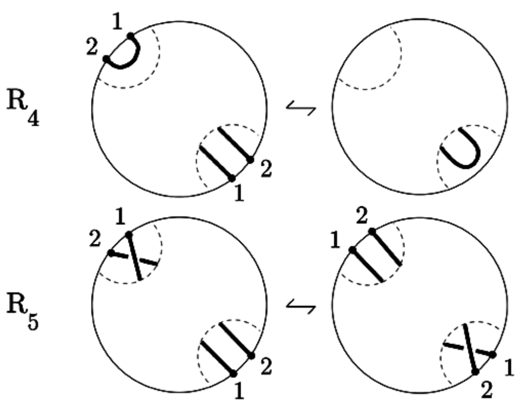

Fig. 8. Generalized Reidemeister moves for projective space

Theorem 12. [3] *Two links L_0 and L_1 in the the projective space are equivalent if and only if their diagrams can be joined by a finite sequence of generalized Reidemeister moves $R_1, \dots, R_5$ and diagram isotopies.*

Proof. It is easy to see that each Reidemeister move produces equivalent links, hence a finite sequence of Reidemeister moves and isotopies on a diagram does not change the equivalence class of the link.

On the other hand, if we have two equivalent links L_0 and L_1, then we have an ambient isotopy, namely: $H : \mathbb{RP}^3 \times [0,1] \to \mathbb{RP}^3$, such that $l_1 = h_1 \circ l_0$. At each $t \in [0,1]$ we have a link L_t, defined by $h_t(l_0)$. As for links in $\mathbf{S}^3$, using general position theory we can assume that the projection of L_t is not regular only a finite number of times, and that at each of these times it violates only one condition.

From each type of violation a transformation of the diagram appears, that is to say, a generalized Reidemeister move, as it follows (see Figures 5 and 9).

So diagrams of two equivalent links can be joined by a finite sequence of generalized Reidemeister moves $R_1, \dots, R_5$ and diagram isotopies. □

– from violations V_1, V_2 and V_3 we obtain the classic Reidemeister moves R_1, R_2 and R_3;
– from violation V_4 we obtain move R_4;
– from violation V_5 we obtain move R_5.

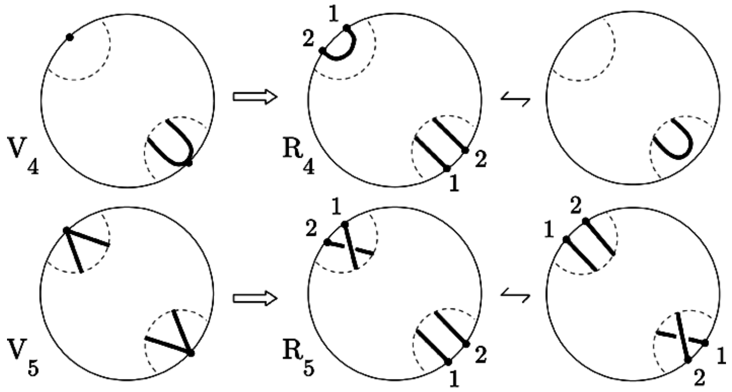

Fig. 9. Regularity violations produce generalized Reidemeister moves

4. Links in $L(p,q)$

4.1. Diagrams

We improve the definition of diagram for links in lens spaces given by Gonzato [4]. We can assume $p > 2$, since we have already seen in the previous sections the particular cases $L(1,0) \cong \mathbf{S}^3$ and $L(2,1) \cong \mathbb{RP}^3$. Consider the construction of the lens space $L(p,q) = B^3/\sim$ we give in the preliminaries, where F is the quotient map.

Definition 13. Let L be a link in $L(p,q)$. Consider $L' := F^{-1}(L)$; by moving L via a small isotopy in $L(p,q)$, we can suppose that:

i) L' does not meet the poles S and N of B^3;

ii) $L' \cap \partial B^3$ consists of a finite set of points.

So L' is the disjoint union of closed curves in $\text{int}B^3$ and arcs properly embedded in B^3 (i.e. only the boundary points belong to ∂B^3).

Let $\mathbf{p} : B^3 \smallsetminus \{N,S\} \to B_0^2$ be the projection defined in the following way: take $x \in B^3 \smallsetminus \{N,S\}$, construct the circle (or the line) $c(x)$ through N, x and S and set $\mathbf{p}(x) := c(x) \cap B_0^2$.

Take L' and project it using $\mathbf{p}_{|L'} : L' \to B_0^2$. For $P \in \mathbf{p}(L')$, $\mathbf{p}_{|L'}^{-1}(P)$ may contain more than one point; in this case, we say that P is a *multiple point*. In particular, if it contains exactly two points, we say that P is a *double point*. We can assume, by moving L via a small isotopy, that the projection $\mathbf{p}_{|L'} : L' \to B_0^2$ of L is *regular*, namely:

1) the arcs of the projection contain no cusps;

2) the arcs of the projection are not tangent to each other;

3) the set of multiple points is finite, and all of them are actually double points;

4) the arcs of the projection are not tangent to ∂B_0^2;

5) no double point is on ∂B_0^2;

6) $L' \cap \partial B_0^2 = \emptyset$.

Now let Q be a double point and consider $\mathbf{p}_{|L'}^{-1}(Q) = \{P_1, P_2\}$. We suppose that P_2 is nearer to S than P_1. Take U as an open neighborhood of P_2 in L' such that $\mathbf{p}(\overline{U})$ does not contain other double points and does not meet ∂B_0^2. We call U *underpass*. Take the complementary set in L' of all the underpasses. Every connected component of this set (as well as its projection in B_0^2) is called *overpass*. The underpasses are visualized in the projection by removing U from L' before projecting the link (see Figure 10).

Definition 14. A *diagram* of a link L in $L(p,q)$ is a regular projection of $L' = F^{-1}(L)$ on the equatorial disk B_0^2, with specified overpasses and underpasses.

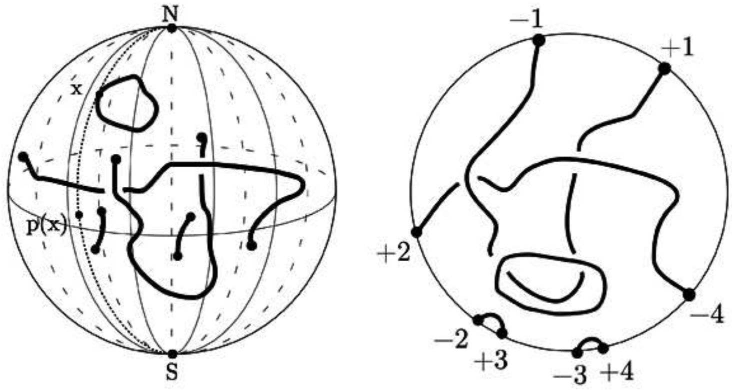

Fig. 10. A link in $L(9,1)$ and corresponding diagram

We label the boundary points of the link projection, in order to show the identifications. Assume that the equator is oriented counterclockwise if we look at it from N. Consider the t endpoints of the overpasses that come from arcs of L' that are above the equator. Label them $+1,\ldots,+t$ according to the orientation of ∂B^2_0. Then label the other t points on the boundary, that come from arcs of L' under the equator, as $-1,\ldots,-t$, where for each $i=1,\ldots,t$, we have $+i \sim -i$. An example is shown in Figure 10.

We want to explain which diagram violations arise from condition 1)-6). For conditions 1), 2) and 3) we already know that the corresponding violations are V_1, V_2 and V_3 of Figure 2.

Condition 4), as in the projective case, has a corresponding violation V_4. On the contrary, condition 5) does not behave as in the projective case. Indeed two diagrammatic violations arise from it (V_5 and V_6), as Figure 11 shows. The difference between the two violations is that V_5 involves two arcs of L' that end in the same hemisphere of ∂B^3, on the contrary V_6 involves arcs that end in different hemispheres.

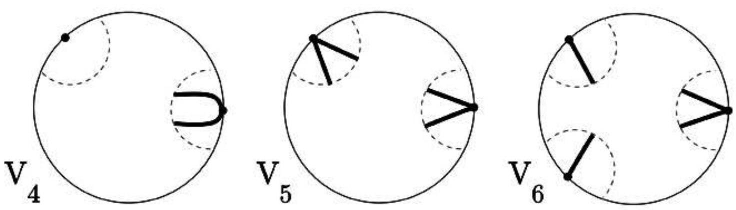

Fig. 11. Violations V_4, V_5 and V_6

Finally condition 6) produces a family of violations called $V_{7,1},\ldots,$ $V_{7,p-1}$ (see Figure 12). The difference between them is that $V_{7,1}$ has the arcs of the projection identified directly by $g_{p,q}$, while $V_{7,k}$ has the arcs identified by $g^k_{p,q}$, for $k=2,\ldots,p-1$.

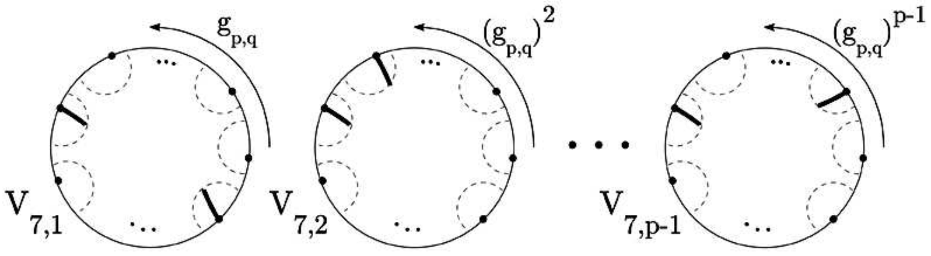

Fig. 12.Violations $V_{7,1}, V_{7,2},\ldots,V_{7,p-1}$

It is easy to see what kind of small isotopy on L is necessary, in order to make the projection of the link regular when we deal with violations $V_1,\ldots,V_6$. Now we explain how the link can avoid to meet ∂B^2_0 up to isotopy, that is to say, avoid $V_{7,1},\ldots,V_{7,p-1}$.

We start with a link with two arcs that ends on ∂B^2_0. If we suppose that the endpoints of the arcs are connected by $g_{p,q}$, (a $V_{7,1}$ violation), then we can label the endpoints B and C, in a way such that $C=g_{p,q}(B)$. In this case the required isotopy is the one that lift up a bit the arc ending in B and lower down the other one.

Now if we suppose that the endpoints of the arcs are connected by a power of $g_{p,q}$, (a $V_{7,k}$ violation with $k > 1$), then we can label the points B and C such that $C = g^k_{p,q}(B)$. In this case the required isotopy is similar to the one of the example in $L(9,1)$ of Figure 13. In lens spaces with $q \neq 1$, the new arcs end in the faces specified by the map $f_3 \circ g_{p,q}$.

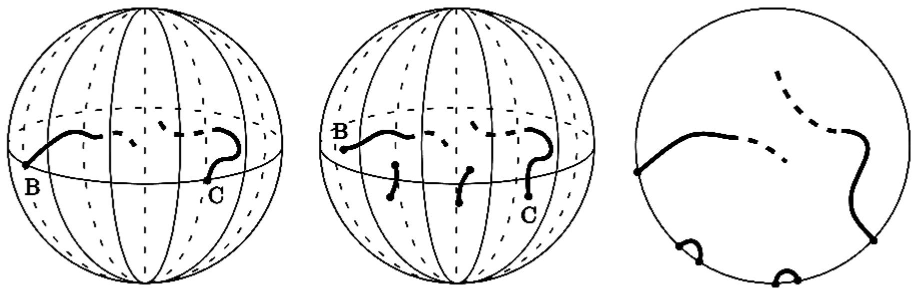

Fig. 13.Avoiding ∂B^2_0 in $L(9,1)$

4.2. Generalized Reidemeister moves

Again, with the aim of discovering when two diagrams represent equivalent links in $L(p,q)$, we generalize Reidemeister moves for diagrams of links.

Definition 15. The *generalized Reidemeister moves* on a diagram of a link $L \subset L(p,q)$ are the moves R_1, R_2, R_3 of Figure 4 and the moves R_4, R_5, R_6 and R_7 of Figure 14.

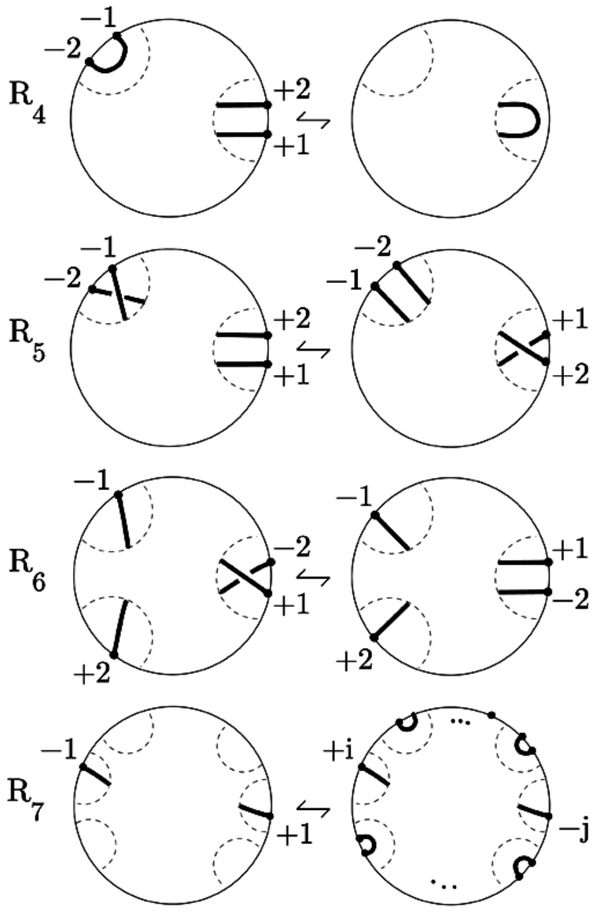

Fig. 14. Generalized Reidemeister moves

Theorem 16. *Two links L_0 and L_1 in $L(p,q)$ are equivalent if and only if their diagrams can be joined by a finite sequence of generalized Reidemeister moves $R_1, \ldots, R_7$ and diagram isotopies.*

Proof. It is easy to see that each Reidemeister move produces equivalent links, hence a finite sequence of Reidemeister moves and isotopies on a diagram does not change the equivalence class of the link.

On the other hand, if we have two equivalent links L_0 and L_1, then we have an ambient isotopy between the two ambient spaces, namely: $H : L(p,q) \times [0,1] \to L(p,q)$. At each $t \in [0,1]$ we have a link L_t, defined by $h_t(l_0)$. Again, as for links in $\mathbf{S}^3$, using general position theory we can assume that the projection $\mathbf{p}(L'_t)$ is not regular only a finite number of times, and that at each of these times it violates only one condition.

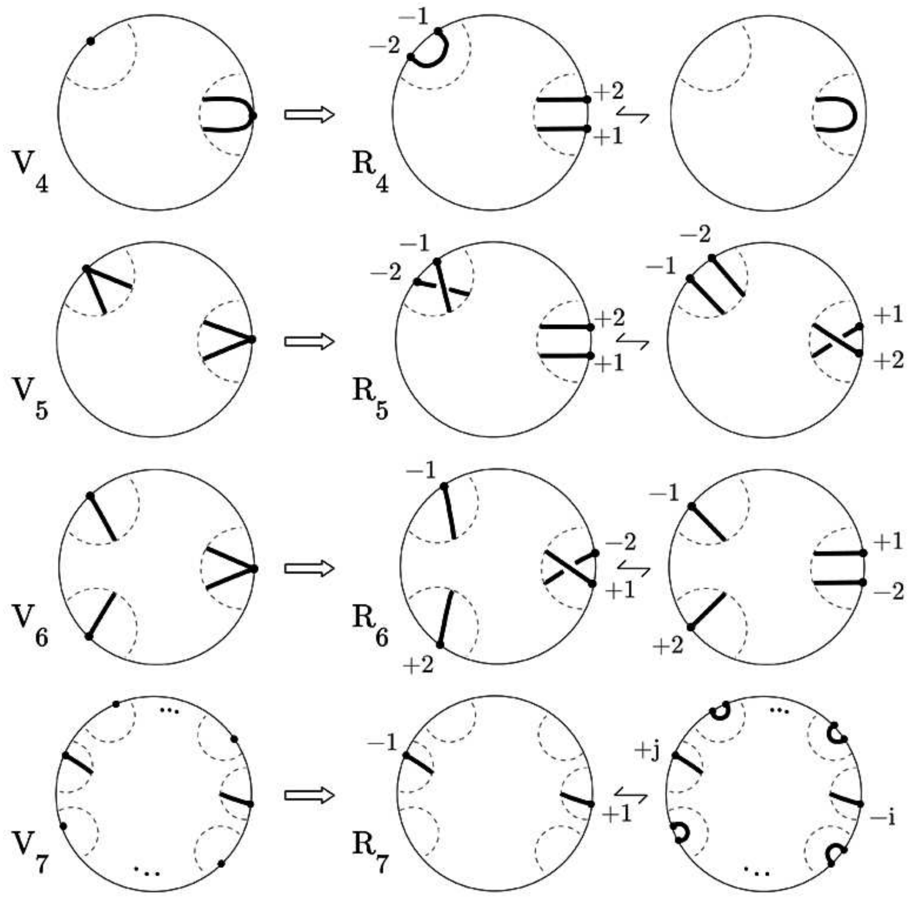

Fig. 15. Regularity violations produce generalized Reidemeister moves

From each type of violation a transformation of the diagram appears, that is to say, a generalized Reidemeister move, as it follows (see Figures 5 and 15):

- from violations V_1, V_2 and V_3 we obtain the classic Reidemeister moves R_1, R_2 and R_3;
- from violation V_4 we obtain move R_4;
- from violation V_5, we obtain two different moves, if the arcs L' with endpoints on the boundary are from the same side with respect to equator, then we obtain R_5, on the contrary we obtain R_6;
- for condition 6 we have a family of violation $V_{7,1}, \ldots, V_{7,p-1}$, from which we obtain the moves $R_{7,1}, \ldots, R_{7,p-1}$.

Indeed, if an arc cross the equator during the isotopy, then we have a class of moves, $R_{7,1} = R_7, R_{7,2}, \ldots, R_{7,p-1}$. All these moves can be seen as the composition of R_7, R_6, R_4 and R_1 moves. More precisely, the move $R_{7,k}$ with $k = 2, \ldots, p-1$, can be obtained by the following sequence of moves: first we perform an R_7 move on one overpass that end on the equator and the corresponding point in a small arc, then we repeat for $k-1$ times the three moves R_6-R_4-R_1 necessary to retract the small arc with same sign ending point (see an example in Figure 16).

So we can exclude $R_{7,2}, \ldots, R_{7,p-1}$ from the move set and keep only $R_{7,1} = R_7$. As a consequence, any pair of diagrams of two equivalent links can be joined by a finite sequence of generalized Reidemeister moves $R_1, \ldots, R_7$ and diagram isotopies. □

References

[1] Brody, E. J. *The topological classification of the lens spaces* / E. J. Brody // Ann. of Math. – 1960. – Vol. **71**. – P. 163 – 184.

[2] Burde, G. *Knots* / G. Burde, H. Zieschang – Walter de Gruyter. – Berlin; New York, 2003.

[3] Drobotukhina, Y. V. *An analogue of the Jones polynomial for links in* $\mathbb{RP}^3$ *and a generalization of the Kauffman-Murasugi theorem* / Y. V. Drobotukhina // Leningrad Math. J. – 1991. – Vol. **2**. – P. 613 – 630.

[4] Gonzato, M. *Invarianti polinomiali per link in spazi lenticolari*/ M. Gonzato // Degree thesis. – University of Bologna, 2007.

[5] Manfredi, E. *Fundamental group of knots and links in lens spaces*/ E. Manfredi // Degree thesis. – University of Trieste, 2010.

[6] Prasolov, V. V. *Knots, links, braids and 3-manifolds. An introduction to the new invariants in low-dimensional topology*/ V. V. Prasolov, A. B. Sossinsky // Transl. of Math. Monographs. – Vol. **154**. – Amer. Math. Soc.: Providence, RI, 1997.

[7] Roseman, D. *Elementary moves for higher dimensional knots* / D. Roseman // Fund. Math. – 2004. – Vol. **184**. – P. 291 – 310.

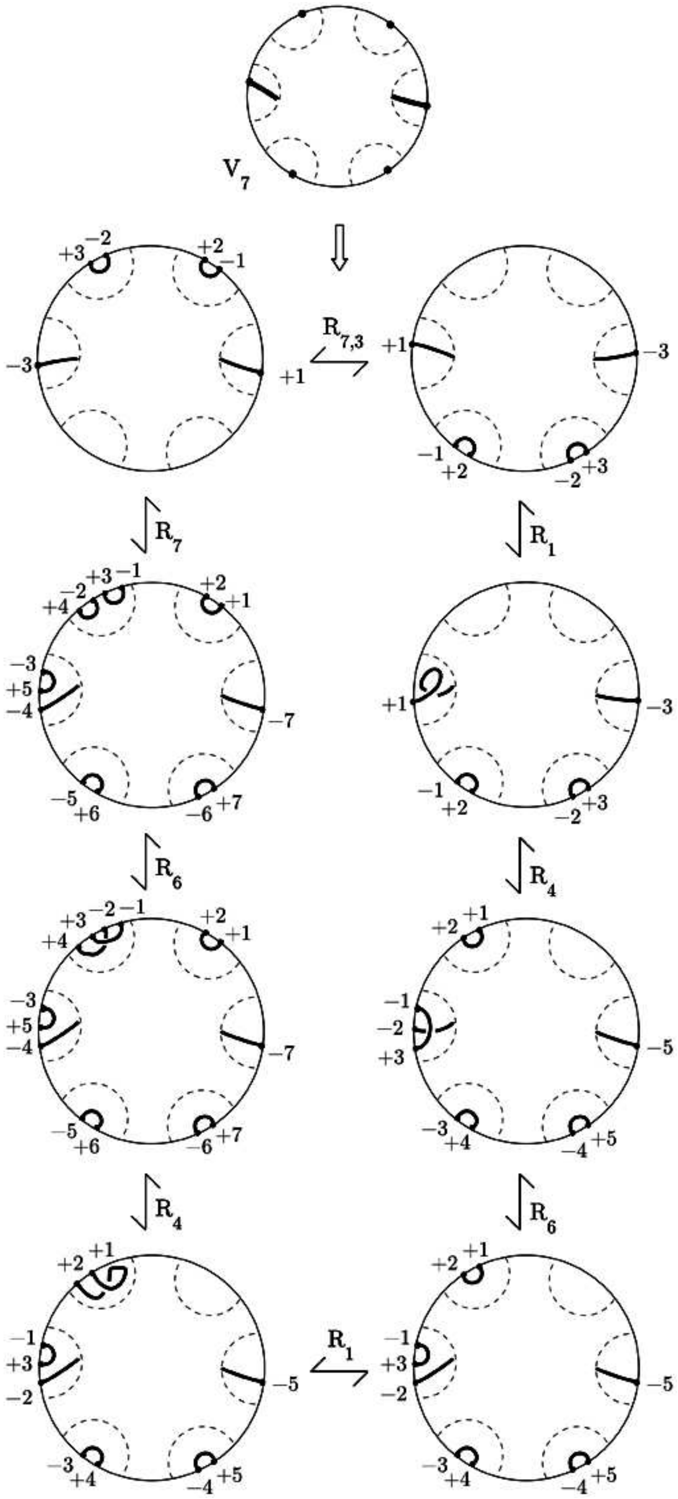

Fig. 16. How to reduce a composite move